2.3.P3 Let $A \in M_n(\mathbf{R})$. Explain why the non-real eigenvalues of A (if any) must occur in conjugate pairs.

2.3.P4 Consider the family $\mathcal{F} = \left\{ \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \right\}$ and show that the hypothesis of commutativity in (2.3.3), while sufficient to imply simultaneous unitary upper triangularizability of $\mathcal{F}$, is not necessary.

2.3.P5 Let $\mathcal{F} = \{A_1, \ldots, A_k\} \subset M_n$ be a given family, and let $\mathcal{G} = \{A_i A_j : i, j = 1, 2, \ldots, k\}$ be the family of all pairwise products of matrices in $\mathcal{F}$. If $\mathcal{G}$ is commutative, it is known that $\mathcal{F}$ can be simultaneously unitarily upper triangularized if and only if every eigenvalue of every commutator $A_i A_j - A_j A_i$ is zero. Show that assuming commutativity of $\mathcal{G}$ is a weaker hypothesis than assuming commutativity of $\mathcal{F}$. Show that the family $\mathcal{F}$ in (2.3.P4) has a corresponding $\mathcal{G}$ that is commutative and that it also satisfies the zero eigenvalue condition.

2.3.P6 Let $A, B \in M_n$ be given, and suppose A and B are simultaneously similar to upper triangular matrices; that is, $S^{-1}AS$ and $S^{-1}BS$ are both upper triangular for some nonsingular $S \in M_n$. Show that every eigenvalue of $AB - BA$ must be zero.

2.3.P7 If a given $A \in M_n$ can be written as $A = Q\Delta Q^T$, in which $Q \in M_n$ is complex orthogonal and $\Delta \in M_n$ is upper triangular, show that A has at least one eigenvector $x \in \mathbf{C}^n$ such that $x^T x \neq 0$. Consider $A = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$ to show that not every $A \in M_n$ can be upper triangularized by a complex orthogonal similarity.

2.3.P8 Let $Q \in M_n$ be complex orthogonal, and suppose that $x \in \mathbf{C}^n$ is an eigenvector of Q associated with an eigenvalue $\lambda \neq \pm 1$. Show that $x^T x = 0$. See (2.1.P8a) for an example of a family of 2-by-2 complex orthogonal matrices with both eigenvalues different from ± 1. Show that none of these matrices can be reduced to upper triangular form by complex orthogonal similarity.

2.3.P9 Let $\lambda, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $A \in M_n$, suppose that x is a nonzero vector such that $Ax = \lambda x$, and let $y \in \mathbf{C}^n$ and $\alpha \in \mathbf{C}$ be given. Provide details for the following argument to show that the eigenvalues of the bordered matrix $\mathcal{A} = \begin{bmatrix} \alpha & y^* \\ x & A \end{bmatrix} \in M_{n+1}$ are the two eigenvalues of $\begin{bmatrix} \alpha & y^*x \\ 1 & \lambda \end{bmatrix}$ together with $\lambda_2, \ldots, \lambda_n$: Form a unitary U whose first column is $x/\|x\|_2$, let $V = [1] \oplus U$, and show that $V^*\mathcal{A}V = \begin{bmatrix} B & \bigstar \\ 0 & C \end{bmatrix}$, in which $B = \begin{bmatrix} \alpha & y^*x/\|x\|_2 \\ \|x\|_2 & \lambda \end{bmatrix} \in M_2$ and $C \in M_{n-2}$ has eigenvalues $\lambda_2, \ldots, \lambda_n$. Consider a similarity of B via $\operatorname{diag}(1, \|x\|_2^{-1})$. If $y \perp x$, conclude that the eigenvalues of $\mathcal{A}$ are $\alpha, \lambda, \lambda_2, \ldots, \lambda_n$. Explain why the eigenvalues of $\begin{bmatrix} \alpha & y^* \\ x & A \end{bmatrix}$ and $\begin{bmatrix} A & x \\ y^* & \alpha \end{bmatrix}$ are the same.

2.3.P10 Let $A = [a_{ij}] \in M_n$ and let $c = \max\{|a_{ij}| : 1 \leq i, j \leq n\}$. Show that $|\det A| \leq c^n n^{n/2}$ in two ways: (a) Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A. Use the arithmetic-geometric mean inequality and (2.3.2a) to explain why $|\det A|^2 = |\lambda_1 \cdots \lambda_n|^2 \leq ((|\lambda_1|^2 + \cdots + |\lambda_n|^2)/n)^n \leq (\sum_{i,j=1}^n |a_{ij}|^2/n)^n \leq (nc^2)^n$. (b) Use Hadamard's inequality in (2.1.P23).

2.3.P11 Use (2.3.1) to prove that if all the eigenvalues of $A \in M_n$ are zero, then $A^n = 0$.

2.3.P12 Let $A \in M_n$, let $\lambda_1, \ldots, \lambda_n$ be its eigenvalues, and let $r \in \{1, \ldots, n\}$. (a) Use (2.3.1) to show that the eigenvalues of the compound matrix $C_r(A)$ are the $\binom{n}{r}$ possible products $\lambda_{i_1} \cdots \lambda_{i_r}$ such that $1 \leq i_1 < i_2 < \cdots < i_r \leq n$. (b) Explain why $\operatorname{tr} C_r(A) = S_r(\lambda_1, \ldots, \lambda_n) = E_r(A)$; see (1.2.14) and (1.2.16). (c) If the eigenvalues of